Subjective Probability as Modality

(Exposition of Gardenfors, "Qualitative probability as an intensional logic", *J. Philos. Logic* 1975, with some notes on its sequel)

1. Probability is not extensional

Even if all and only the creatures with hearts are creatures with kidneys it does not follow that *it seems likely to me*, or to anyone, that a given creature with a heart also has kidneys, let alone that all creatures with hearts do so. So the phrase *it seems likely to me* needs to be classified, in logical terms, as non-extensional. That still leaves two possibilities: it may intensional or hyperintensional.

If we could infer that necessarily equivalent possibilities (in whatever sense of necessity you like) have the same status for me as far as my subjective grading is concerned, then *it seems likely to me* could be classified as intensional -- if not, as hyperintensional. Given our possibly illogical states of opinion, the latter may be the more realistic option. But for restricted purposes, it may be possible to model opinion as intensional.

2. Opinion as subjective grading of possible alternatives

One function of opinion is surely to assess, divide, and sort out the various possible outcomes of our actions, in the various possible states of nature compatible with all that we know and believe. The latter form a subset of all possible worlds, the region of possibility that we give probability 1; and the sorting out of its subsets can be thought of as an assignment of different probabilities to them.

If we try to model this view by the means usually employed in logic, we will formulate sentences that describe this, with appropriate truth conditions for them. The choices we need to make will include how informative we make these sentences, how rich the logical resources in the language, and whether we will limit the description to a single agent ("me") or to multiple agents.

3. Gardenfors' 1975 scheme

As basic expression of opinion Gardenfors allows the qualitative judgement of form "It seems at least as likely [to me] that A as that B", symbolized as "A $\geq$B". The language contains atomic sentences that can describe any state of affairs at all, the absurd (necessarily false) sentence 0, and the usual propositional connectives with their standard meaning.

> I'll use ~ for **not**, & for **and**, v for **or**, $\supset$ for the material **if .. then**,
> and $\equiv$ for the material **if and only if**.
>
> Some other abbreviations are useful:
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> "1" for "(0$\supset$ 0)",
>
> "A < B" for "~(A $\geq$ B)", "A > B" for "B < A", etc.;
>
> "A $\approx$ B" for "(A $\geq$ B) & (B $\geq$ A)".
>
> Finally, with an eye to the connections with modal logic we can introduce:
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> "□A" for "A $\approx$ 1" "◊A" for "A > 0"

4. Clues to the logic -- Law of Small Numbers

The notation and the intuitive meaning quickly suggest transitivity and reflexivity for $\geq$, as well of course as axioms or theorems to the effect that 1 $\geq$ A, for any sentence A. But what will convey the distinctively numerical grading that lies behind this?

The clue for how to bring that into the logic lies in De Finetti's neat little Law of Small Numbers. Given n sentences A_1, ..., An, how many of them will be true? If we equate the truth values **true** and **false** with values 1 and 0 then the answer to that question is precisely: *the sum of their truth values*.

This can still of course be any number from 0 to n; let us call that number $\#\{A_1, ..., A_n\}$. What is my expected value for this number? Well, it must accordingly be my expectation value for the function which is the sum of their truth values. Expectation is linear, so that is the sum of my expectation value for each of these sentences taken individually. However, my expectation value for a single sentence (proposition, event) is simply my subjective probability for it. So we have proved:

Expectation value for $\#\{ A_1, ..., A_n \}$ *= Sum of the probabilities of* $A_1, ..., A_n$

That is the law. It connects probability with expected relative frequency in the small.

5. Importing the law into logic

How can we bring this insight into the logic? We can find an expression for the statement that just as many of $\{ A_1, ..., A_n \}$ as of $\{ B_1, ..., B_n \}$ are true. To see this, intuitively anyway, begin with n=1:

$\#\{A\} = \#\{B\}$ iff $A \equiv B$ is true

$\#\{1, A\} = \#\{B, C\}$ iff (A&B&C) v (~A&B&~C) v(~A& ~B&C)

and so forth. I won't give the general definition here, but just assume that it has been done, so now the left hand side expressions can be regarded as in the language, defined by the right hand side.

Hence in the probabilistic context we can introduce a statement that says that such numeral equality is certain. Gardenfors' symbol for that is an emphatic version of the letter E:

" $A_1 ...A_n$ **E** $B_1 ... B_n$ " for "$[\#\{ A_1, ..., A_n \}= \#\{ B_1, ..., B_n \}] \approx 1$"

The crucial axiom is now that if this numerical equality holds, and the probabilities of the first n-1 As are all at least as high as those of the corresponding Bs, then the last B will make up for that:

If A1...An **E** B1...Bn and A1 $\geq$ B1, ..., $A_{(n-1)} \geq B_{(n-1)}$ then Bn $\geq$ An

6. One-person, multi-state models

We imagine the possible worlds to each comprise besides a "state of nature' also the subjective probability function (opinion) of our subject in that world. So a model looks like this:

<U, P, V> where U is non-empty (the worlds)

-- for each x in U, P_x is a probability function defined on the subsets of U (the propositions)

-- V assigns to each sentence A the set V(A) of worlds in which A is true

We require V to be as follows:

V(0) is empty

$V(\sim A) = U - V(A)$

$V(A\&B) = V(A) \cap V(B)$ [etc.]

$V(A \geq B) = \{x \text{ in } U\text{: } P_x(V(A)) \geq P_x(V(B))\}$

The logic sketched above, and made precise in Gardenfors' paper as system QP, is sound and complete for this class of models, as well as decidable.

7. Subjective necessity and possibility

Given the meanings we assigned above to □ and ◊ it follows at once that the characteristic D (deontic logic) axiom is valid in this language:

□ A implies ◊ A

In fact, Gardenfors proves that the modal logic D is sound and complete for the fragment of the language that has the Boolean connectives plus these two modal connectives as sole logical resources.

Let's call that the "necessity fragment" of the language. Then we can look for connections with other standard normal modal logics as follows.

Say that world y is possible relative to world x precisely if $P_x(\{y\}) > 0$. In that case the condition that each world is possible relative to itself would validate the characteristic T axiom:

□ A implies A

If we add that the probability function Px is the same for all worlds x, then, not surprisingly, we eliminate most logische Spitzfindigkeit, and arrive at modal logic S5 (where □□ and ◊□ both just amount to □, while □◊ and ◊◊ both just amount to ◊).

8. Beyond Gardenfors: Multi-person, multi-state models

The above scheme can be generalized in several ways: we can allow for multiple agents in those worlds, and we can express the numerical grades explicitly. The basic opinion attributing sentence should then be of form "$L^i_p A$", to be read as "Agent i's subjective probability for A is at least p".

This sentence will be true in a certain set of worlds, which itself can be among the propositions on which these subjective probabilities are defined. Thus these agents can have opinions about others' opinions (as well as about their own), and about how those opinions relate to what is actually the case. The usual name for such models is now *type spaces* (since the individuals are not further described, but only serve to identify types of opinion).

Logics can now be strengthed by means of axioms both concerning introspection (opinions about one's own opinion) and concerning common beliefs and knowledge (identified minimally as true belief to maximal degree). From less to more audacity the introspection (or perhaps we should say 'transparency of mind') axioms one encounters are that the individual knows that s/he is certain about something if s/he is, that s/he knows which propositions s/he is not certain about, and that s/he knows the subjective probability she has for each proposition.

Common belief and common knowledge are now ubiquitous notions in game theory -- a sentence A is a matter of common belief in a given world if each agent believes that A, and believes of every agent that s/he believes that A, and believes of every agent that s/he believes **that**, and so forth.